What are the real numbers?

Part I: A journey from the intuitive notion to a proper definition

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Four fundamental sets of numbers that we learn about in school are the sets $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, and $\mathbb{R}$. The first one is the set of positive whole numbers, also called the naturals, i.e. $\mathbb{N} = \{1, 2, 3, \ldots\}$. The second set, called the integers, contains all whole numbers, both positive and negative, as well as 0, that is $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$. The third one, $\mathbb{Q}$, is the set of all rational numbers, i.e. fractions of the form m/n where m and n are integers and n is non-zero. But what about $\mathbb{R}$? We know this as the set of real numbers, but how do you describe those numbers? The way we usually think of the reals is as points on a line, the "real number line". This line contains all the rationals, but it also contains other numbers, called irrationals. But who are those other numbers? Sure, one can give examples, like $\sqrt{2}$, the number π, or Euler's constant e. But what about the rest? How many are there?

You will probably agree with me that defining the real numbers as "all those numbers on a line" is kind of an obscure definition; it is much less satisfactory than, for example, the definition of the rationals, which are described as fractions of integers. Can we construct the real numbers in a way similar to the way the rationals are constructed from the integers? This is the main topic of this article. But as someone familiar with the poem *Ithaka*[1] will know, it's the journey that matters, not the destination. So our real goal is to enjoy the journey and learn a lot on the way, with the definition of the reals serving as our destination.

Understanding our destination: $\mathbb{Q}$ vs $\mathbb{R}$

Let us first take a step back and think about what we know of the real numbers. Specifically, let us compare the reals with the rationals and see what they have in common and what not.

To begin with, in both of these sets of numbers we can perform the usual arithmetic operations ($+$, $-$, $\cdot$, $/$) and the result is a number *of the same kind*, e.g. when you add, subtract, multiply, or divide two rational numbers, you get a rational, and the same is true for the reals. This is not true for the naturals or the integers, by the way, because division might give you a number that is not a whole number. The mathematical term used for such a set, in which you can perform all four arithmetic operations with the standard properties that we know from school, is a *number field*, or simply a *field*. $\mathbb{Q}$ and $\mathbb{R}$ are thus both fields.

[1] *Ithaka* is a Greek poem written in 1911 by Constantine Cavafy. It makes reference to Ulysse's adventurous journey back home after the Trojan war, which is the topic of the much older poem *Odyssey*.

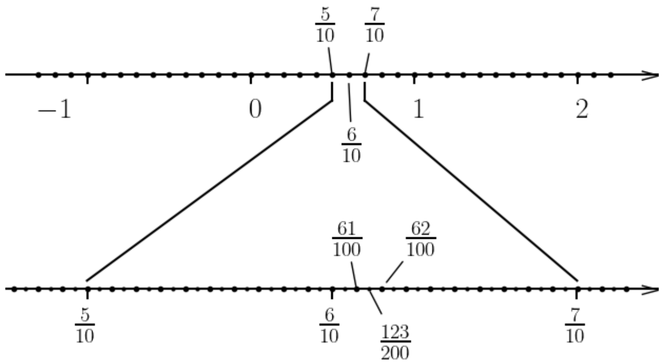

Figure 1: No matter how much we zoom in on the real line, we will always find rationals, and in fact infinitely many of them.

A second property that $\mathbb{Q}$ and $\mathbb{R}$ share is that they have an order relation ("$>$") defined on them that can tell us for any two elements in these sets which one is larger. This is unlike the complex numbers $\mathbb{C}$, for example, which do not come with a natural relation of this kind. Such order relations, when they satisfy the standard properties that we are familiar with (e.g., "if $a > b$ and $b > c$, then $a > c$"), are called **total orders**,[2] hence $\mathbb{Q}$ and $\mathbb{R}$ are both *totally ordered sets*.

Another property that $\mathbb{Q}$ and $\mathbb{R}$ have in common is that, when we put them on a line, they cover the line densely, in the sense that no matter how close you look, there are infinitely many of them (see fig. 1). Given our understanding of $\mathbb{R}$ as "all numbers on a line", this is not saying anything new. But regarding $\mathbb{Q}$, it is a property worth noting: there are infinitely many rationals and, in addition to that, no matter how much you zoom in on the line, you will still find infinitely many of them; there are no "gaps", in the sense that there is no interval (segment) that is free of rationals. This is not true for the naturals $\mathbb{N}$ or integers $\mathbb{Z}$.

Think: Does the set of all fractions of integers whose denominators are powers of 10 (e.g. numbers like $7/10$, $-3/100$, $173/10$, $-35/1$, $1831213/10000$, etc.) cover densely the real line? Is this set the same as $\mathbb{Q}$ or is it a smaller set? (In other words, are there rational numbers that cannot be written this way?)

Leaving point gaps

But here is one difference between $\mathbb{Q}$ and $\mathbb{R}$: while $\mathbb{Q}$ leaves no *interval* gaps, it does leave *point* gaps. For example, there is no rational at the point $\sqrt{2}$ (see the Appendix for a proof). Does $\mathbb{R}$ leave gaps? Given our understanding of $\mathbb{R}$ as the set of all points on a line, it shouldn't leave gaps by definition. But as already pointed out, this definition is not satisfactory. We hope that when we give a proper definition of the reals we can show that it doesn't leave gaps.

Why is this important? Does it really matter whether a set of numbers "leaves gaps"

[2] The specification *total* here means that for any two (distinct) elements, one will be larger than the other. In contrast, the relation $A \subset B$ that compares two sets is a *partial order*, because sometimes neither $A \subset B$ nor $B \subset A$ is true.

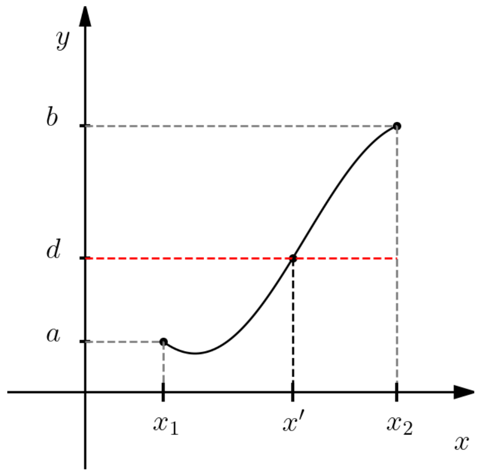

Figure 2: The Intermediate Value Theorem states that if a continuous function defined on an interval $[x_1, x_2] \subset \mathbb{R}$ takes the values $a = f(x_1)$ and $b = f(x_2)$ at the endpoints, then it must also take every value d between a and b at some $x' \in (x_1, x_2)$. In other words, the graph of the function must meet exactly the horizontal line $y = d$ (it cannot "skip" values). This theorem depends on the properties of the real numbers; it does not hold if f is defined on the rationals.

or not? It turns out that it does. Several fundamental theorems of calculus depend on this. It's likely that you are already familiar with some of these theorems, like the Intermediate Value Theorem, which says that if a continuous function f defined on the real numbers attains two values a and b (i.e. $f(x_1) = a$ for some x_1 and $f(x_2) = b$ for some x_2), then it also attains every intermediate value between a and b, at some x between x_1 and x_2. See fig. 2. Is this true if f is defined on the rationals? No, take for example the function with formula $f(x) = x^2 - 2$. We have $f(0) = -2$ and $f(2) = 2$, but there is no *rational* number x such that $f(x) = 0$. The intermediate value theorem is thus not satisfied if we work with functions defined on the rationals, exactly because $\mathbb{Q}$ leaves gaps that allow the function to "skip" some values.[3]

Planning our route: Why not decimal expansions?

One may try to define the real numbers by way of decimal expansions, i.e. by saying that the real numbers are all whole or decimal numbers, including those with infinite decimal places, whether the decimal digits repeat or not.[4] But how do we know that these are "all" the numbers? In other words, how do we know that if we put all those numbers on a line, they don't leave gaps like the rationals do? Perhaps you are inclined to believe that there can't be any other numbers, because the decimal expansions provide arbitrary precision. But this is also true for the rationals, as explained above, hence it is

[3] Another example of a theorem that depends on $\mathbb{R}$ not leaving gaps is the extreme value theorem, which says that every continuous function defined on a closed interval $[a, b]$ attains a maximum as well as a minimum value. This is again not true if the function is defined only on rational points in $[a, b]$.

[4] Recall that the rationals are those whole or decimal numbers that either have a finite number of decimal places or their decimal digits repeat indefinitely after some point.

not enough to guarantee that there can't be any other numbers.

The above discussion raises the question: How can we *ever* be sure that there are no other numbers? What does it even mean that there are gaps left, unless we have already defined more numbers? The fact that the intermediate value theorem can be proven for functions defined on the reals but fails if we restrict ourselves to the rationals shows that this is not just a philosophical discussion. Below we will make precise what we mean by "leaves gaps", and then we'll give a definition of the reals that provably doesn't leave gaps. Although it *is* possible to do this through decimal expansions, it is a route that we are not going to take. Instead, we will follow a different route, one that will take us through more interesting places and that will teach us more. Remember that the goal is not just to reach the destination, but also to enjoy the journey and become a little bit wiser.

Choosing our ship: Geometric intuition vs Set-theoretic definitions

In mathematics we often use geometric thinking to get intuition. This is especially true in the field of mathematics that studies the properties of real numbers (which is part of the field of real analysis). However, intuitive understanding isn't enough when we want to be precise. After all, one's intuitive understanding might be different from that of someone else. Just to give an example, consider the question of what the length of a point is. Some people will say that it has zero length, while others might raise an objection and argue as follows: a line segment is made up of points and it has non-zero length; how can something of non-zero length be the result of putting together things of zero length?

Today mathematicians agree that, although geometric thinking can be very useful in understanding certain concepts, our definitions and arguments must not make any reference to geometric intuition. Instead, they must be expressed in the language of set theory (and logic). This means expressions like "x is an element of the set S" ($x \in S$), or "for each element y in the set S..." ($\forall y \in S$), and so on. Anything that does not already have a well-defined meaning must be properly defined (in this language) before we can use it.

When we said above that $\mathbb{Q}$ "leaves gaps", we were invoking our geometric intuition. We'd like to have a proper definition of what it means for a set to "leave gaps", using the language of sets. This will be the bulk of our journey. However, before going into the open sea, let us first make a practice expedition to get to know our boat.

A practice expedition: Expressing properties in set theory language

To illustrate how one can turn an intuitive notion into a rigorous definition, we will use an example that will also turn out to be very relevant later on, so this is training that is bound to prove useful. Consider the following question: what is the difference between closed intervals, i.e. intervals of the form $[a, b]$, and half-open intervals of the form $[a, b)$? Geometrically, intervals of the first type contain their right endpoint, while intervals of the second type don't. How can we express this without making any reference to the geometry we associate with these intervals (i.e. as parts of the real line)? Can we define it without using terms like "the right endpoint"?

It turns out that it is not hard to do so, once we observe that, apart from being segments of the real line, these intervals are sets with an order relation ("$>$") and, in the case of $[a, b]$, the right endpoint b turns out to be its largest element. What is the largest element of $[a, b)$? There is none; no matter how close to b we choose a number, there is always a larger number contained in the interval. We are not allowed to choose b itself, because it is not a member of the set. As a result, no element of $[a, b)$ has the property that it is larger than all other elements.

We have thus found a property that distinguishes between the two types of intervals: the first has a largest element, while the second doesn't. Is the property "has a largest element" a precise statement? Yes, as the following definition, expressed in the language of set theory, shows:

Definition 1. *A totally ordered set S has a largest element if there exists some $x \in S$, such that $x \geq y$ for all $y \in S$.*[5]

This definition makes precise what we mean by "contains a largest element". In particular, it makes no reference to intuitive/geometric notions. It does make reference to an order relation ("$\geq$") and this is the reason that we require S to be a *totally ordered* set. Total orders are things that can be defined in the language of sets, but we are not going to do this here. One doesn't want to get lost venturing forever in side quests; it is an expedition best left for another day.

Now that we are equipped with the above definition, we may say that a property that distinguishes a set of the form $[a, b]$ from a set of the form $[a, b)$ is that the former has a largest element while the latter doesn't.

Think: Does the set of numbers $\{0, 0.9, 0.99, 0.999, \ldots\}$ have a largest element? Does the set of natural numbers $\mathbb{N}$ have a largest element? Does a *finite* subset of $\mathbb{R}$ (i.e. one containing finitely many elements) always have a largest element?

In the open sea: Defining "leaves gaps" in the language of sets

Now that we have already seen an example of how to turn a geometric property into a statement in the language of sets, we'd like to do the same with the main property that distinguishes the rationals from the real numbers, i.e. the fact that $\mathbb{Q}$ leaves (point) gaps on the line. How can we express this in terms of sets and order relations?

If we had defined $\mathbb{R}$ and identified it with the real line, our job would have been easy: $\mathbb{Q}$ leaves gaps simply means that there are elements of $\mathbb{R}$ that do not belong to $\mathbb{Q}$. But remember that we haven't defined $\mathbb{R}$ yet,[6] so our task will be to express the property "$\mathbb{Q}$ leaves gaps", in a precise way, without making reference to a larger set. Our first stop will be the notion of *cuts*.

Cuts in $\mathbb{Q}$

Suppose that we want to split the rationals into two sets, one that includes all those numbers that are smaller than 1 and those that are larger than 1. We would have to decide where 1 would go; let's say it goes with the larger ones. We therefore have the

[5] We write "$\geq$" instead of "$>$", i.e. we allow x to be equal to y, because they might happen to be the same element.

[6] And we will in fact need the concepts that we develop here in order to define $\mathbb{R}$, so we can't just postpone this problem until after we have defined it.

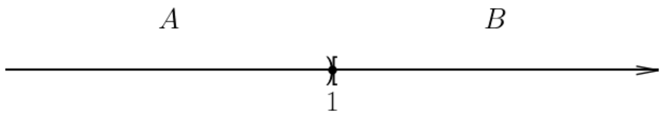

Figure 3: A cut in $\mathbb{Q}$. The sets A and B only contain rational numbers. They are disjoint and taken together they make up $\mathbb{Q}$.

sets $A = \{x \in \mathbb{Q} : x < 1\}$ and $B = \{x \in \mathbb{Q} : x \geq 1\}$. See fig. 3. This pair of sets has the following properties:

(i) A and B are both non-empty.

(ii) $A \cap B = \emptyset$ (A and B have no elements in common).

(iii) $A \cup B = \mathbb{Q}$ (Together A and B make up all of $\mathbb{Q}$).

(iv) Every element of A is smaller than every element of B.

(v) A has no largest element.

Such a pair of sets will be called a **cut** (because it's as if we "cut" the number line at a certain point, e.g. at 1) and will be written as $A|B$. As you may imagine, we can get different cuts by "'cutting" at different points, i.e. by substituting the number 1 in the definition of A and B by something else. In fact, any rational number q provides a cut if we define $A = \{x \in \mathbb{Q} : x < q\}$ and $B = \{x \in \mathbb{Q} : x \geq q\}$. But are all cuts obtained this way or are there more cuts than rational numbers?

Think: In order to better understand a definition, it is often useful to think how its essence would change if we modified some part of it. For example, what is the importance of property (iv) above? To answer this, do the following exercise: Find a way to split the rationals into two sets such that properties (i), (ii), (iii), and (v) are satisfied, but property (iv) is not.

No smallest element for B

Note that by definition of a cut, it is the first set, A, that must have no largest element. We will stick to this convention. Regarding the set B, there is no requirement of either having or not having a smallest element. But is it possible that (in addition to A having no largest element) B has no smallest element? Can you find such a cut? Remember that the rest of the properties in the definition of a cut must also be satisfied (don't forget property (iii)!).

If you answered that it is impossible, you gave up too fast. Consider the following cut $C|D$ in $\mathbb{Q}$: C contains all negative rationals and all those rationals whose square is less than 2. D contains all other positive rationals. Written with symbols, we have

$$\begin{aligned} C &= \{x \in \mathbb{Q} : x < 0 \text{ or } x^2 < 2\} \\ D &= \text{all the rest of } \mathbb{Q}. \end{aligned} \tag{1}$$

What are the elements in D? By definition they are all positive rational numbers whose square is *at least* 2. But given that there is no rational whose square is exactly 2,

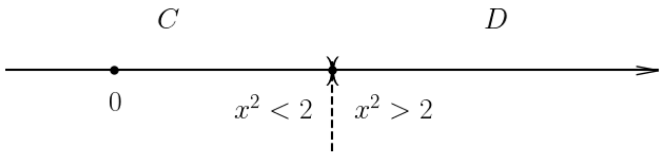

Figure 4: $C|D$ is a cut in $\mathbb{Q}$. C contains all negative rationals and all those rationals whose square is less than 2. D contains all positive rationals whose square is larger than 2. Since there is no rational whose square is *equal* to 2, C and D together make up all of $\mathbb{Q}$.

we may as well say that D contains only those positive rationals whose square is *larger* than 2, i.e. $D = \{x \in \mathbb{Q} : x > 0 \text{ and } x^2 > 2\}$. See fig. 4.

Is $C|D$ a cut in $\mathbb{Q}$? Yes, it is easy to check that it satisfies the five necessary properties mentioned above: (i) C and D are non-empty ii) they are disjoint (have no elements in common), (iii) together they form all of $\mathbb{Q}$, (iv) every element of C is smaller than every element of D, and (v) C contains no largest element. But this time, in addition to C having no largest element, D doesn't have a smallest element. How is this possible? Exactly because $\mathbb{Q}$ leaves point gaps! If it didn't, then wherever we chose to "cut" the line, there would be a number. Since that number can't go into the set C (because C can't have a largest element), it would have to go into D and be its smallest element. But because $\mathbb{Q}$ leaves gaps on the line, it is possible to cut at a point where there is no (rational) number and thus D has no smallest element.

Concluding

We have just found a property, expressed in the language of sets, which is satisfied only by those sets that we intuitively understand as "leaving gaps". This can thus serve as our definition. It does not apply only to rationals, but we can express it for any totally ordered set.

Definition 2. *A totally ordered set S* ***leaves gaps*** *if it is possible to find two sets $A, B \subset S$, such that they form a cut in S, and moreover B has no smallest element.*

Of course, by "a cut in S" we mean the same as a cut in $\mathbb{Q}$, but now A, B are subsets of S and when taken together they must give S (i.e. $A \cup B = S$) instead of $\mathbb{Q}$.

Think: Why do we have to restrict the definition to totally ordered sets? Which parts of the definition make use of the order relation?

Seagulls in the sky

We have come all the way across the ocean, so we must be closing in on our destination. Let us look around and see whether there are any hints.

Real numbers as cuts in $\mathbb{Q}$

Intuitively, the cut $C|D$ where

$$C = \{x \in \mathbb{Q} : x < 0 \text{ or } x^2 < 2\}, \qquad D = \text{all the rest of } \mathbb{Q}$$

represents the number $\sqrt{2}$, while the cut $A|B$ with

$$A = \{x \in \mathbb{Q} : x < 1\}, \qquad B = \text{all the rest of } \mathbb{Q}$$

represents the number 1. One should be able to do the same with any point on the line, i.e. every real number. We already saw that this is the case with every rational number q: just define the cut $A_q|B_q$ where $A_q = \{x \in \mathbb{Q} : x < q\}$ and $B_q =$ all the rest of $\mathbb{Q}$. This cut represents q.

Can we do the same with irrational numbers? As already mentioned, intuitively it should be possible to cut the line at any real number. However, we cannot just write $A = \{x \in \mathbb{Q} : x < r\}$, if r is not a rational number. Why? Because we don't know what r is yet, so how can we compare x to r? This is why in constructing the cut $C|D$ above, we wrote $x^2 < 2$ in the definition of C, rather than $x < \sqrt{2}$. The expression $x^2 < 2$ is perfectly valid in the realm or rational numbers.

Let's reflect a bit more on the cut $C|D$ above. What we did with this cut is that we created something that has the "feeling" of the number $\sqrt{2}$, which is an irrational number, using only expressions that involved rationals. Perhaps this is the way to go if we want to "construct" the real numbers from stuff that we already know: we could *define* $\sqrt{2}$ to be the cut $C|D$ constructed above. It looks like there is land on the horizon!

How can we generalize this to other irrational numbers? We can certainly do something similar with other radicals, e.g. $\sqrt[3]{5}$ can be represented by the cut with A-part $\{x \in \mathbb{Q} : x^3 < 5\}$.[7] A similar approach also applies to logarithms, e.g. $\log_{10} 3$ can be represented by the cut with A-part $\{x \in \mathbb{Q} : 10^x < 3\}$. With a little more work we can represent many other irrational numbers, including π and Euler's constant e. But we will never know whether these are *all* irrational numbers or we are still missing some. We will never know if we have filled in all the holes this way.[8]

In order to make sure we fill in all the holes, we are going to take a more radical approach: we will forego giving explicit expressions for each number separately, and instead we will say that the real numbers are *all possible cuts* in $\mathbb{Q}$, whether we can come up with an explicit expression for them or not. That is:

Definition 3. *A* ***real number*** *is a cut in* $\mathbb{Q}$.

We denote by $\mathbb{R}$ the set of all real numbers (cuts in $\mathbb{Q}$). A drawback of this definition is that it doesn't give us a good overview of what numbers we have just created. We will remedy this by *studying* our creation. This is very common in mathematics: first define something, then study the properties that it has. For example, recall when you had first defined the derivative at x_0 as $f'(x_0) = \lim\limits_{x \to x_0} \frac{f(x)-f(x_0)}{x-x_0}$. You didn't know at the time that the derivative would obey rules like the product rule; you deduced such properties afterwards.

Our goal below is to show that the set of real numbers thus defined has all the nice properties that we want it to have. We are not going to do this in detail, but we will sketch the ideas.

[7]Note that with cubic roots we may write simply $x^3 < 5$ instead of "$x < 0$ or $x^3 < 5$"; all negatives have the property that $x^3 < 5$, hence they are included in this expression.

[8]It turns out that it is *impossible* to give an explicit expression for every real number. This is another expedition best left for another day.

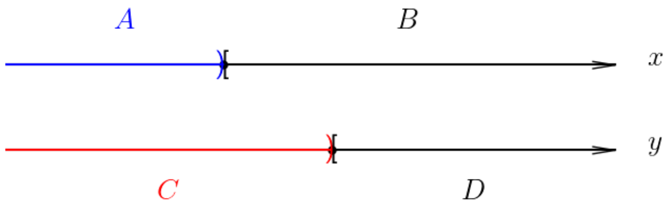

Figure 5: Intuitively, a real number (cut) $y = C|D$ is larger than or equal to the real number $x = A|B$ if y "cuts" relatively to the right compared to x. This can be rigorously expressed by saying $A \subset C$.

Is this Ithaka?

We have defined $\mathbb{R}$ in a way that made sense, but does the set thus defined have the properties that we anticipate? Let's look at those properties one by one.

$\mathbb{R}$ contains $\mathbb{Q}$

A basic property that the real numbers must satisfy is that they extend $\mathbb{Q}$, i.e. that $\mathbb{Q}$ is their subset. However, the definition that we have given makes the real numbers things of a different nature. They are not just numbers like the rationals are; they are pairs of sets (cuts). In this sense, $\mathbb{Q}$ is not a subset of the newly defined $\mathbb{R}$. But as we have already seen, each member of $\mathbb{Q}$ can be *identified* with a corresponding member of $\mathbb{R}$ in a very straightforward way: the number $q \in \mathbb{Q}$ can be identified with the cut $A_q|B_q$ where $A_q = \{x \in \mathbb{Q} : x < q\}$ and $B_q = \{x \in \mathbb{Q} : x \geq q\}$, i.e. "the cut at q". The cut $A_q|B_q$ belongs to $\mathbb{R}$, so, in this sense, the elements of $\mathbb{Q}$ can also be thought of as elements of $\mathbb{R}$, making $\mathbb{Q}$ essentially a subset of $\mathbb{R}$. We will use the notation q^* to denote the cut (member of $\mathbb{R}$) that corresponds to the rational number q. We emphasize that while q is a member of $\mathbb{Q}$ ($q \in \mathbb{Q}$), q^* is a member of $\mathbb{R}$ ($q^* \in \mathbb{R}$).

$\mathbb{R}$ is a totally ordered set

Another important property that this newly created $\mathbb{R}$ must satisfy is to be a totally ordered set. Otherwise, it would be impossible to picture it as a "line" of numbers. Although we have defined what the members of $\mathbb{R}$ are, we haven't specified how to order them. But it is easy to do so. The intuition is the following: Since any real number is a "cut" at a certain point of the real line, we say that y is larger than x if y "cuts" at a point relatively to the right of x. To make this rigorous, note that if y cuts further to the right, then the A-part of x will be a subset of the A-part of y. See fig. 5. We thus define:

Definition 4. *For any real numbers $x = A|B$ and $y = C|D$, we say that x is less than or equal to y ($x \leq y$) if $A \subset C$.*[9]

Note how the above definition of "$x \leq y$" makes use of the definition of x and y as cuts (pairs of subsets) and avoids any reference to geometric notions, such as "x lies to the left

[9]Here and in the rest of this article we use the symbol $\subset$ to mean "subset or equal" (some authors use $\subseteq$ instead). If we want to say that a set is a proper subset of another set (i.e. certainly not equal), we may write $\subsetneq$.

of y". It is not hard to check that this definition satisfies all standard properties of a total order, which we are not going to list here.[10] One property worth noting, however, is the fact that when comparing two numbers of the form p^* and q^* (i.e. two real numbers that represent rationals), then this order relation coincides with the standard order relation in $\mathbb{Q}$, i.e. $p^* < q^*$ if and only if $p < q$.

Arithmetic operations

How do you add two cuts $A|B$ and $C|D$? If we want to call these cuts "real numbers", we'd better find a way to add them!

To start with something easy, let's say we want to add the numbers 3^* and 4^*, which are the members of $\mathbb{R}$ that we identify with the rationals 3 and 4, respectively. Clearly, the result of adding them *must* be 7^*, i.e. $(3+4)^*$. But we are not going to simply define it to be $(3+4)^*$; this method would only work when adding real numbers that represent rationals. We want to get this result in a way that also works for real numbers that do not have rational counterparts.

Let us consider the A-parts of these numbers: the A-part of 3^* is $A_3 = \{x \in \mathbb{Q} : x < 3\}$, that of 4^* is $A_4 = \{x \in \mathbb{Q} : x < 4\}$, and that of 7^* is $A_7 = \{x \in \mathbb{Q} : x < 7\}$. How does A_7 relate to A_3 and A_4? Simple: A_7 contains all those numbers that one can get by adding a number in A_3 and a number in A_4. That is,

$$A_7 = \{s + t : s \in A_3, t \in A_4\}. \tag{2}$$

The right-hand side can be read as "the set of all those numbers that can be obtained by adding two numbers s and t, where s is any member of A_3 and t is any member of A_4". Let's verify that this set contains exactly the same elements as A_7. First observe that if s is less than 3 and t is less than 4, then their sum must be less than 7. Hence, any member of this set is contained in A_7. Conversely, any number that is less than 7 can be written as a sum of two numbers, the first of which is less than 3 and the second less than 4 (for example, $6.99 = 2.995 + 3.995$). This makes sure that every element of A_7 is also contained in the set on the right-hand side of the above equation. We conclude that these two sets contain exactly the same elements, i.e. eq. (2) holds.

Although when adding $3^* + 4^*$ we already knew the result that we were looking for, the above discussion suggests a general definition that can work for all real numbers:

Definition 5. *The sum $x + y$ of two real numbers $x = A|B$ and $y = C|D$ is the real number $z = E|F$, where*

$$\begin{aligned} E &= \{s + t : s \in A, t \in C\} \quad \textit{and} \\ F &= \textit{all the rest of } \mathbb{Q}. \end{aligned} \tag{3}$$

It is important to note that the addition $s + t$ appearing in the definition of E is an addition of *rational* numbers, since A and C are subsets of $\mathbb{Q}$. Hence our definition is not a circular one; we make use of addition of rationals in order to define addition of real numbers.

[10] Let us give an example of how one could prove those properties: Is it true that $x \le y$ and $y \le z$ implies $x \le z$? Yes. Write $x = A|B$, $y = C|D$, and $z = E|F$. The first two inequalities mean that $A \subset C$ and $C \subset E$, respectively. We can thus deduce that $A \subset E$, which by definition means that $x \le z$ and this completes the proof.

Let's look at an example. What is $\sqrt[3]{5}+\sqrt[3]{7}$? Recall that $\sqrt[3]{5}$ is a real number whose A-part is $\{x \in \mathbb{Q} : x^3 < 5\}$ and $\sqrt[3]{7}$ is a real number whose A-part is $\{x \in \mathbb{Q} : x^3 < 7\}$. Therefore, according to the above definition, $\sqrt[3]{5}+\sqrt[3]{7}$ is a real number whose A-part is the set of all numbers that can be obtained by adding any two rationals s and t that satisfy $s^3 < 5$ and $t^3 < 7$.

One can define analogously other arithmetic operations. The idea is always that one uses the corresponding operations of $\mathbb{Q}$ and the definition of real numbers as cuts in $\mathbb{Q}$ in order to define the arithmetic operations in $\mathbb{R}$. The operations thus defined satisfy all standard properties, such as transitivity and commutativity (for addition and multiplication), distributivity of multiplication over addition, etc. Moreover, these operations behave as they should with respect to the total order that we have defined earlier, i.e. if $x > y$ and $a > 0^*$, then $a \cdot x > a \cdot y$. Finally, when the newly defined arithmetic operations are applied to members of $\mathbb{R}$ that represent rationals, they coincide with the corresponding operations on rationals, i.e. $p^* \cdot q^* = (p \cdot q)^*$.

$\mathbb{R}$ doesn't leave gaps

We created the real numbers by considering all cuts in $\mathbb{Q}$. This way $\mathbb{R}$ included all members of $\mathbb{Q}$, but also all gaps left by $\mathbb{Q}$ on the number line. As such, the real numbers shouldn't leave any gaps, right? But what if instead of cuts in $\mathbb{Q}$ we consider cuts in $\mathbb{R}$? What if, by introducing the real numbers, we have *changed* the number line, making it more dense and introducing new points that are not covered?

To make the above precise, the question is: does $\mathbb{R}$ leave gaps, according to definition 2? In other words, is it possible to find two subsets U and V of $\mathbb{R}$, such that they satisfy the five properties of a cut in $\mathbb{R}$ and additionally V has no smallest element?

The answer is no, but this will have to wait until our next journey to be shown. There is only so much one can do in a trip! In that second journey we are going to talk about another very important property of $\mathbb{R}$, which is the *least upper bound property*. It will then easily follow from the least upper bound property that $\mathbb{R}$ doesn't leave gaps. Of course, the destination will only be the end; we will make sure to enjoy that journey as well!

Acknowlegdements

The formal part of the exposition follows Charles C. Pugh's treatment of the topic in his book Real Mathematical Analysis [1] (in particular Section 1.2). If you'd like to take a deep dive in the waters we have crossed, I highly recommend this book.

References

[1] Charles Chapman Pugh. *Real Mathematical Analysis*. Springer, 2002.

Appendix: Proof that $\sqrt{2}$ is not a rational number

The need to define a larger set of numbers than the rationals arose from the fact that there are quantities that come up naturally in mathematics which cannot be expressed as ratios of integers. But how do we know that a number like $\sqrt{2}$ or π cannot be expressed as a ratio of integers? Although today we are told so in school, this is not at all obvious. The first number to be shown not to be rational was probably $\sqrt{2}$, and the proof dates back at least to the Ancient Greeks. Here we give a modern proof that relies on simple algebraic manipulations and some straightforward facts about odd and even numbers.

Theorem 6. *There is no rational number whose square is equal to* 2.

Proof. Suppose that there is a rational number, i.e. a fraction with integer numerator and denominator, whose square equals 2. As a first step, we simplify this fraction. Specifically, if both numerator and denominator are even numbers, then we divide both by 2. We repeat this until at least one of them is not even, so we cannot simplify further in this way (specifically by dividing by 2). We end up with a fraction, say $\frac{a}{b}$, where either a or b is an odd number.

Because the above process does not change the value of the fraction, we have that $\left(\frac{a}{b}\right)^2 = 2$, or equivalently $a^2 = 2b^2$. Now, the fact that a^2 is twice another integer (b^2) means that it is necessarily even. But only even numbers can have even squares, hence a is itself even.

Now rewrite the equation $a^2 = 2b^2$ as

$$2 \cdot \left(\frac{a}{2}\right)^2 = b^2. \tag{4}$$

Given that a is even, $a/2$ is integer, hence the left hand side is twice an integer quantity, i.e. an even number. Because this is equal to b^2, it implies that b is also an even number. This contradicts the fact that not both a and b can be even numbers at the same time.

□